16. $y = \log(Ax)$ ∴ {1 const.}

Diff wrt x

$$\frac{dy}{dx} = \frac{A}{Ax}$$

$$\therefore x\frac{dy}{dx} = 1$$

17. $r\theta = C_1 \cos\omega\theta + C_2 \sin\omega\theta$ (2 const)

Diff wrt θ

$$r + \theta\frac{dr}{d\theta} = -C_1\omega\sin\omega\theta + C_2\omega\cos\omega\theta$$

Diff wrt θ

$$\frac{dr}{d\theta} + \frac{dr}{d\theta} + \theta\frac{d^2r}{d\theta^2} = -C_1\omega^2\cos\omega\theta - C_2\omega^2\sin\omega\theta$$

$$= -\omega^2(C_1\cos\omega\theta + C_2\sin\omega\theta)$$

$$\therefore 2\frac{dr}{d\theta} + \theta\frac{d^2r}{d\theta^2} = -\omega^2 r\theta$$

$$\therefore \theta\frac{d^2r}{d\theta^2} + 2\frac{dr}{d\theta} + \omega^2 r\theta = 0$$

Eq of 1st Order & Degree

Eg 1

$$\frac{dy}{dx} = e^{x-2y}$$

$$\frac{dy}{dx} = e^x \cdot e^{-2y}$$

$$\frac{1}{e^{-2y}}\,dy = e^x\,dx$$

$$e^{2y}dy - e^x dx = 0$$

Integrating both sides

$$\int e^{2y}dy - \int e^x dx = c$$

$$\frac{e^{2y}}{2} - e^x = c$$

~~Eg 2~~ $\cancel{3e^x \tan y\,dy + (1-e^x)\sec^2 y\,dy = 0}$

Methods to solve D.E :-

I. Seperation of variables
II. Homogeneous eq. in x & y
III. Non-homogeneous eq. in x & y
IV. Exact D.E. V using I.F.
VI. Linear eq
VII. Substitution to reduce to above forms

SEPERATION OF VARIABLES:-

Method I: If an eq. can be reduced to the form -

$$f_1(x)\,dx + f_2(y)\,dy = 0$$

Where $f_1(x)$ is a funct of x & $f_2(y)$ is a funct. of y then, solution can be obtained by

$$\int f_1(x)\,dx + \int f_2(y)\,dy = c$$

Eg 1 ahead back

Eg 2

$$3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$$

$\div$ by $\tan y\,(1-e^x)$

$$\frac{3e^x}{1-e^x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

Int. both sides

$$\int \frac{3e^x}{1-e^x}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = c$$

Let $1-e^x = u$; $\tan y = v$

$\therefore -e^x dx = du$; $\sec^2 y\,dy = dv$

$$\therefore \int \frac{-3\,du}{u} + \int \frac{dv}{v} = c$$

$$\therefore -3\log u + \log v = \log c$$

$$\therefore -3\log(1-e^x) + \log \tan y = \log c$$

$$\therefore \log\left(\frac{\tan y}{(1-e^x)^3}\right) = \log c$$

$$\therefore \tan y = c\,(1-e^x)^3$$

Tel: 9820246760 / 9769479368 / 9820009639

navlakhi.com navlakhi.mobi navlakhi.tv

Eg II-A

1 $\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$

$y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$

$y - ay^2 = x\frac{dy}{dx} + a\frac{dy}{dx}$

$y - ay^2 = (x+a)\frac{dy}{dx}$

$(y - ay^2)\,dx - (x+a)\,dy = 0$

$\div$ by $(x+a)(y-ay^2)$

$\frac{dx}{x+a} - \frac{dy}{y-ay^2} = 0$

Int. both sides

$\int \frac{dx}{x+a} + \int \frac{dy}{ay^2 - y + \frac{1}{4a} - \frac{1}{4a}} = c$

$LT = \frac{MT^2}{4FT} = \frac{y^2}{4a}$

$\int \frac{dx}{x+a} + \int \frac{dy}{\left(\sqrt{a}y - \frac{1}{2\sqrt{a}}\right)^2 - \left(\frac{1}{2\sqrt{a}}\right)^2} = c$

Let $\sqrt{a}\,y - \frac{1}{2\sqrt{a}} = t \qquad \sqrt{a}\,dy = dt$

$\int \frac{dx}{x+a} + \frac{1}{\sqrt{a}} \int \frac{dt}{t^2 - \left(\frac{1}{2\sqrt{a}}\right)^2} = c$

$\log(x+a) + \frac{1}{\sqrt{a}} \times \frac{2\sqrt{a}}{2} \log\left|\frac{t - \frac{1}{2\sqrt{a}}}{t + \frac{1}{2\sqrt{a}}}\right| = \log c$

$\log(x+a) + \log\left|\frac{\sqrt{a}\,y - 1/\sqrt{a}}{\sqrt{a}\,y}\right| = \log c$

$\log(x+a) + \log\left|\frac{ay - 1}{ay}\right| = \log c$

$(x+a)(ay-1) = cay$

$(x+a)(1-ay) = (-ca)y = c'y$

2. $a\left(x\frac{dy}{dx}+2y\right)=2xy\frac{dy}{dx}$

$ax\frac{dy}{dx}+2ay=2xy\frac{dy}{dx}$

$2ay+(ax-2xy)\frac{dy}{dx}=0$

$2ay\,dx+(ax-2xy)dy=0$

$2ay\,dx+x(a-2y)dy=0$

$\div$ by xy

$2a\frac{dx}{x}+\left(\frac{a-2y}{y}\right)dy=0$

Int.

$\int\frac{2a}{x}dx+\int\left(\frac{a}{y}-2\right)dy=c$

$\Rightarrow 2a\log x+a\log y-2y=\log c$

$\div$ by a

$\log x^2y-\log c'=\frac{2y}{a}$

$\boxed{(x^2y)^a=ce^{2y}}$

$x^2y=ce^{2y/a}$

3. $(x+1)\frac{dy}{dx}+1=2e^{-y}$

$(x+1)\frac{dy}{dx}=2e^{-y}-1$

$\int\frac{dy}{2e^{-y}-1}-\int\frac{dx}{x+1}=c$

$\int\left(\frac{e^y}{2-e^y}\right)dy-\int\frac{dx}{x+1}=c$

$-\log(2-e^y)-\log(x+1)=-\log c'$

$\log(2-e^y)+\log(x+1)=\log c'$

$\boxed{(x+1)(2-e^y)=c''}$

OR $(x+1)(-e^y)+2x+2=c''$

$2x+2-c''=(x+1)e^y$

$\therefore (x+1)e^y=2x+c$

Tel: 9820246760 / 9769479368 / 9820009639

4 $x\cos x\cos y + \sin y\dfrac{dy}{dx} = 0$

$x\cos x\cos y\,dx + \sin y\,dy = 0$

$\div \cos y$ & Int.

$\int x\cos x\,dx + \int \tan y\,dy = c$

$\boxed{\int uv = u\int v\,dx - \int u'\left(\int v\,dx\right)dx}$

$x\int \cos x\,dx - \int 1\cdot\int \cos x\,dx + \int \tan y\,dy = c$

$\int \tan y = \log \sec y$

$x\sin x - \int \sin x\,dx + \log(\sec y) = c$

$x\sin x + \cos x + \log(\sec y) = c$

OR $x\sin x + \cos x - \log(\cos y) = c$

5 $x(1-y)\,dx + (1+y^2)(x-1)\,dy = 0$

$\div (x-1)(1-y)$

$\int \dfrac{x}{x-1}dx + \int \dfrac{1+y^2}{1-y}dy = c$

$\log(x-1) + \int \dfrac{1}{1-y}dy + \int \dfrac{y^2\,dy}{1-y} = c$

$\log(x-1) - \log(1-y) + \int \dfrac{y\cdot y}{1-y}dy = c$

Let $1-y = t \quad \therefore -dy = dt$ & $y = 1-t$

$\log(x-1) - \log(1-y) + \int \dfrac{(1-t)}{t}(-dt) = c$

$\log\left(\dfrac{x-1}{1-y}\right) - \int\left(\dfrac{1}{t} - 1\right)dt = c$

$\log\left(\dfrac{x-1}{1-y}\right) - \log t - t = \log c$

$\log\left(\dfrac{x-1}{1-y}\right) - \log(1-y) - 1 + y = \log c$

Tel: 9820246760 / 9769479368 / 9820009639

5. $x(1-y)dx + (1+y^2)(x-1)dy = 0$

$\div$ by $(1-y)(x-1)$

$\therefore \int \frac{x}{x-1}dx + \int \frac{1+y^2}{1-y}dy = c$

Let $x-1 = u$, $x = 1+u$; $1-y = v$; $y = 1-v$

$\therefore dx = du$; $-dy = dv$

$\int \frac{(1+u)du}{u} + \int \frac{dy}{1-y} - \int \frac{(1-v)^2 dv}{v} = c$

$\int \frac{du}{u} + \int du + \int \frac{dy}{1-y} - \int \frac{dv}{v} + 2\int dv - \int v\,dv = c$

$\log u + u - \log(1-y) - \log v + 2v - \frac{v^2}{2} = c$

$\log(x-1) + (x-1) - \log(1-y) - \log(1-y) + 2(1-y) - \frac{(1-y)^2}{2} = c$

$x + \log(x-1) - 2\log(1-y) - 2y + \frac{-1+y-\frac{y^2}{2}}{} + (1-c) = 0$

$= x + \log(x-1) - \log(1-y)^2 - y - \frac{y^2}{2} + c = 0$

6. $\frac{y}{x}\frac{dy}{dx} = \sqrt{1+x^2+y^2+x^2y^2}$

$\frac{y}{x}\frac{dy}{dx} = \sqrt{1+x^2+y^2(1+x^2)}$

$\frac{y}{x}\frac{dy}{dx} = \sqrt{(1+x^2)(1+y^2)}$

$\frac{y}{\sqrt{1+y^2}}dy = x\sqrt{1+x^2}\,dx$

$2\int \frac{y}{\sqrt{1+y^2}}dy - 2\int x\sqrt{1+x^2}\,dx = \cancel{\log} c$

Let $1+y^2 = u$; $1+x^2 = v$

$\therefore 2y\,dy = du$; $\therefore 2x\,dx = dv$

$\int \frac{du}{\sqrt{u}} - \int \sqrt{v}\,dv = c$

$\frac{u^{1/2}}{1/2} - \frac{v^{3/2}}{3/2} = c$ $\therefore \sqrt{u} - \frac{v^{3/2}}{3} = 2c$

$\sqrt{1+y^2} - \frac{1}{3}(1+x^2)^{3/2} = 2c$

7

$$\frac{dy}{dx} = \frac{x\sin x}{2e^y \sinh y}$$

$$\frac{dy}{dx} = \frac{x\sin x}{\cancel{2}e^y\left(\frac{e^y - e^{-y}}{\cancel{2}}\right)}$$

$$e^y(e^y - e^{-y})\,dy = x\sin x\,dx$$

$$\int(e^{2y} - 1)\,dy - \int x\sin x\,dx = c$$

$$\frac{e^{2y}}{2} - y - \left[-x\cos x + \int\cos x\,dx\right] = c$$

$$\frac{e^{2y}}{2} - y + x\cos x - \sin x = c$$

[METHOD II DO AFTER METHOD IV]

GOTO METHOD III: Homogeneous eq

Method IV: ~~Homogeneous Eqs:~~ Exact D.E.:

Eq. of the form $M\,dx + N\,dy = 0$ is exact if $M\,dx + N\,dy = du$

$\therefore M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}.dy$

$\therefore M = \frac{\partial u}{\partial x}$; $N = \frac{\partial u}{\partial y}$

$\therefore$ Condition for exactness: $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ (Each $= \frac{\partial^2 u}{\partial x \partial y}$)

Solution of exactness: $\boxed{\underset{y\text{ const.}}{\int M\,dx} + \underset{\text{no } x \text{ terms}}{\int N\,dy} = c}$

Eg 1 ~~$x^2 y\,dx - (x^3+y^3)dy = 0$~~

Eg 1 $(2x^2+6xy-y^2)dx + (3x^2-2xy+y^2)dy = 0$

is of the form

$M\,dx + N\,dy = 0$

$\frac{\partial M}{\partial y} = 6x - 2y$; $\frac{\partial N}{\partial x} = 6x-2y$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ The eq. is exact

Therefore, its G.S. is

$\int M\,dx + \int (N \text{ terms not containing } x)\,dy = c$

$\int_{y\ \text{const.}} (2x^2+6xy-y^2)\,dx + \int y^2\,dy = c$

no x terms

$\therefore 2\frac{x^3}{3} + 6y\frac{x^2}{2} - xy^2 + \frac{y^3}{3} = c$

$\therefore 2x^3 + 9x^2y - 3xy^2 + y^3 = 3c$

Eg II-D

1. $(a^2-2xy-y^2)dx - (x+y)^2 dy = 0$

is of the form $M\,dx + N\,dy = 0$ where $M =$, $N =$

$\frac{\partial M}{\partial y} = -2x - 2y$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-x^2-2xy-y^2) = -2x-2y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Eq is exact.

Hence, the G.S. is

$\int_{y\ \text{const.}} M\,dx + \int N\,dy = c$

no x terms

$\int_{y\ \text{const.}} (a^2-2xy-y^2)\,dx + \int -y^2\,dy = c$

no x terms

$\therefore a^2x - \frac{2x^2y}{2} - xy^2 - \frac{y^3}{3} = c$

2. $(2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x})dy + (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)dx = 0$

$N\,dy + M\,dx = 0$ where $M =$, $N =$

$$\frac{\partial M}{\partial y} = 12x^2 + 4xy - 12y^2 + 2e^{2x} - e^y$$

$$\frac{\partial N}{\partial x} = 4xy + 12x^2 - 12y^2 - e^y + 2e^{2x}$$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Eq. is exact

$\int M\,dx + \int N\,dy = c$

y const. y terms not const

$$4\cdot 12y\frac{x^3}{3} + 2y^2\frac{x^2}{2} + 4\frac{x^4}{4} - 4xy^3 + 2y\frac{e^{2x}}{2}$$

$$- xe^y + 3\frac{y^3}{3} = c$$

$$x^2y^2 + 4x^3y - 4xy^3 + y^3 - xe^y + ye^{2x} + x^4 = c$$

3. $(x - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

form $M\,dx + N\,dy = 0$

where $M =$, $N =$

$$\frac{\partial M}{\partial y} = -4x - 4y \; ; \; \frac{\partial N}{\partial x} = -4y - 4x$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Eq. is exact

Hence, the G.S is

$\int M\,dx + \int N\,dy = c$

y const. y terms & const

$$\int (x - 4xy - 2y^2)dx + \int y^2\,dy = c$$

$$\therefore \frac{x^2}{2} - 4y\frac{x^2}{2} - 2xy^2 + \frac{y^3}{3} = c$$

$$3x^2 - 12x^2y - 12xy^2 + 2y^3 = 6c$$

4. $(x^2+y^2-a^2)x\,dx + (x^2-y^2-b^2)y\,dy = 0$

form $M\,dx + N\,dy = 0$

where $M = (x^2+y^2-a^2)x$

$= x^3 + xy^2 - a^2x$

$N = x^2y - y^3 - b^2y$

$\frac{\partial M}{\partial y} = 2xy$; $\frac{\partial N}{\partial x} = 2xy$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Eq is exact

$\int M\,dx + \int N\,dy = c$

y const. y terms & const.

$\int (x^3+xy^2-a^2x)\,dx + \int (-y^3 - b^2y) = c$

$\frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{a^2x^2}{2} - \frac{y^4}{4} - \frac{b^2y^2}{2} = c$

$\therefore x^4 - y^4 + 2x^2y^2 - 2a^2x^2 - 2b^2y^2 = 4c$

5. $(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

for $M\,dx + N\,dy = 0$

where $M = 1 + e^{x/y}$

$N = e^{x/y} - e^{x/y}\frac{x}{y}$

$\frac{\partial M}{\partial y} = e^{x/y} \cdot \frac{\partial}{\partial y}\frac{x}{y} = -e^{x/y}\frac{x}{y^2}$

$\frac{\partial N}{\partial x} = e^{x/y}\cdot\frac{1}{y} - \left(e^{x/y}\cdot\frac{1}{y} + \frac{x}{y^2}e^{x/y}\right)$

$= \cancel{e^{x/y}\cdot\frac{1}{y}} - \cancel{e^{x/y}\cdot\frac{1}{y}} - e^{x/y}\frac{x}{y^2}$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Eq. is exact

Hence its G.S. is

$\int M\,dx + \int N\,dy = c$

y const y terms & const

$\int (1+e^{x/y})\,dx + \int 0 = c$

$x + \frac{e^{x/y}}{1/y} = c$ $\therefore x + y e^{x/y} = c$

Tel: 9820246760 / 9769479368 / 9820009639

6. $(y^2 e^{xy^2} + 4x^3)dx + (2xy e^{xy^2} - 3y^2)dy = 0$

is of the form $Mdx + Ndy = 0$

where $M = y^2 e^{xy^2} + 4x^3$

$N = 2xy e^{xy^2} - 3y^2$

$\frac{\partial M}{\partial y} = y^2 \cdot 2xy\, e^{xy^2} + 2y e^{xy^2}$

$\frac{\partial N}{\partial x} = 2xy \cdot y^2 e^{xy^2} + 2y e^{xy^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Eq. is exact

$\int_{y\ const} M dx + \int_{y\ terms\ \&\ const} N dy = c$

$\int (y^2 e^{xy^2} + 4x^3) dx + \int -3y^2 dy = c$

$y^2 \frac{e^{xy^2}}{y^2} + \frac{4x^4}{4} - \frac{3y^3}{3} = c$

$e^{xy^2} + x^4 - y^3 = c$

7. $\frac{dy}{dx} = \frac{1 + y^2 + 3x^2 y}{1 - 2xy - x^3}$

$(1 + y^2 + 3x^2 y)dx - (1 - 2xy - x^3)dy = 0$

is the form $Mdx + Ndy = 0$

where $M = 1 + y^2 + 3x^2 y$

$N = -1 + 2xy + x^3$

$\frac{\partial M}{\partial y} = 2y + 3x^2$; $\frac{\partial N}{\partial x} = 2y + 3x^2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Eq. is exact

$\therefore$ G.S. is

$\int_{y\ const} M dx + \int_{y\ terms\ \&\ const} N dy = c$

$\int (1 + y^2 + 3x^2 y) dx + \int (-1) dy = c$

$x + xy^2 + x^3 y - y = c$

Tel: 9820246760 / 9769479368 / 9820009639

Q $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

$(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0$

is of form $M\,dx + N\,dy = 0$

where $M = y\cos x + \sin y + y$

$N = \sin x + x\cos y + x$

$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$

$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Eq is exact. Hence soln is

$\int M\,dx + \int N\,dy = c$

y const   y term & const

$\int(y\cos x + \sin y + y)dx + \int 0 = c$

$y\sin x + x\sin y + xy = c$

[GO TO METHOD V]

Method II: Homogeneous Eq. in x & y

If an equation

$f_1(x,y)dx + f_2(x,y)dy = 0$

is homogeneous in x & y of same degree then,

$\frac{dy}{dx} = -\frac{f_1(x,y)}{f_2(x,y)}$ — (I)

Substitute,

$y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$ — (II)

Sub. (II) in (I)

$\therefore v + x\frac{dv}{dx} = -\frac{f_1(x,vx)}{f_2(x,vx)}$

and integrate.

Tel: 9820246760 / 9769479368 / 9820009639

navlakhi.com navlakhi.mobi navlakhi.tv

Eg 1

$x^2y\,dx - (x^3+y^3)dy = 0$

Since equation is homogeneous

Let $y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$

$\therefore v + x\frac{dv}{dx} = \frac{x^2 \cdot vx}{x^3+v^3x^3}$

$= \frac{v}{1+v^3}$

$x\frac{dv}{dx} = \frac{v}{1+v^3} - v$

$x\frac{dv}{dx} = \frac{v - v - v^4}{1+v^3} = \frac{-v^4}{1+v^3}$

$\int \frac{1+v^3}{v^4}dv + \int \frac{dx}{x} = c$

$\int v^{-4}dv + \int \frac{dv}{v} + \int \frac{dx}{x} = c$

$\frac{v^{-3}}{-3} + \log v + \log x = \log c'$

$\log xv = \log c' + \frac{v^{-3}}{3}$ or $\log xv - \log c' = \frac{1}{3v^3}$

$\log xv = \log c' + \log e^{v^{-3}/3}$ $\therefore \frac{xv}{c'} = e^{\frac{1}{3v^3}}$

$\therefore xv = c'e^{1/3v^3}$

$xv = c'e^{\frac{1}{3v^3}}$

$\therefore y = c'e^{\frac{x^3}{3y^3}}$ $\{\because v = y/x\}$

$\times x$ II B

1 $(x^2+y^2)dx - 2xy\,dy = 0$

$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

Let $y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2xvx} = \frac{1+v^2}{2v}$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

$$\int \frac{2v}{1-v^2}\,dv - \int \frac{dx}{x} = c$$

Let $1-v^2 = t \quad \therefore -2v\,dv = dt$

$$-\int \frac{dt}{t} - \int \frac{dx}{x} = c$$

$$\log t + \log x = c$$

$$\log(1-v^2) + \log x = c$$

$$\log(1-v^2)x = \log c'$$

$$(1-v^2)x = c'$$

$$\left(1-\frac{y^2}{x^2}\right)x = c'$$

$T-2hr$

$\frac{T-2}{2}$

$$x^2 - y^2 = c'x$$

$$x\frac{dy}{dx} + \frac{y^2}{x} = y$$

$$x\frac{dy}{dx} = y - \frac{y^2}{x}$$

$$x\frac{dy}{dx} = \frac{xy-y^2}{x}$$

$$\frac{dy}{dx} = \frac{xy-y^2}{x^2}$$

$\therefore$ Eq is homogeneous

Let $y = vx$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore v + x\frac{dv}{dx} = \frac{vx^2 - v^2x^2}{x^2} = v - v^2$$

$$x\frac{dv}{dx} = v - v^2 - v = -v^2$$

$$\frac{dv}{-v^2} = \frac{dx}{x}$$

$$\int \frac{dv}{v^2} + \int \frac{dx}{x} = c$$

$$-\frac{1}{v} + \log x = \log c$$

$\log x - \log c = \frac{1}{v}$ $\therefore \log \frac{x}{c} = \frac{1}{v}$

$x = c e^{1/v}$

$\therefore x = c e^{x/y}$

3 $2xy\,dx + (y^2 - x^2)dy = 0$

$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$

$\therefore$ Eq. is homogeneous

Let $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{2xvx}{x^2 - v^2x^2} = \frac{2v}{1-v^2}$

$\therefore x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = \frac{v+v^3}{1-v^2}$

$\frac{1-v^2}{v+v^3}dv = \frac{dx}{x}$

$\int \frac{v^2-1}{v+v^3}dv + \int \frac{dx}{x} = c$

$\int \frac{v^2-1}{v(v^2+1)}dv + \int \frac{dx}{x} = c$

$\therefore I + \log x = c$

$I = \int \frac{v^2-1}{v(v^2+1)}dv$

Let $v = \tan\theta$ $\therefore dv = \sec^2\theta\,d\theta$

$I = \int \frac{\tan^2\theta - 1}{\tan\theta\sec^2\theta} \times \sec^2\theta\,d\theta$

$= \int \tan\theta\,d\theta - \int \cot\theta\,d\theta$

$\log\sec\theta - \log\sin\theta$

$= \log\left(\frac{\sec\theta}{\sin\theta}\right) = \log\left(\frac{1}{\sin\theta\cos\theta}\right)$

$\because \tan\theta = v/1$ $\therefore \sin\theta = v/\sqrt{v^2+1}$; $\cos\theta = 1/\sqrt{v^2+1}$

(By $\perp$ Δ^{le} method)

$\therefore I = \log\left(\frac{v^2+1}{v}\right)$

$v = \tan\theta$ $\left(\frac{\sin 2\theta - \cos 2\theta}{\sin\theta\cos\theta}\right)$ $2\int \frac{\cos 2\theta}{\sin 2\theta}\,d\theta$ $\cot 2\theta\,d\theta$

navlakhi.com navlakhi.mobi navlakhi.tv

Tel: 9820246760 / 9769479368 / 9820009639

$\therefore I + \log x = \log c$

$\therefore \log\left(\frac{v^2+1}{v}\right) + \log x = \log c$

$\log x\left(\frac{v^2+1}{x}\right) = \log c$

$x\left(\frac{v^2+1}{v}\right) = c \quad \because v = y/x$

$\therefore x\left(\frac{y^2+x^2}{xy}\right) = c$

$\therefore x^2+y^2 = cy$ ✓

4) $(x+y)\frac{dy}{dx} + (x-y) = 0$

$\frac{dy}{dx} = \frac{y-x}{y+x}$

Let $y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{vx - x}{vx + x} = \frac{v-1}{v+1}$

$x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v-1-v^2-v}{v+1}$

$x\frac{dv}{dx} + \frac{v^2+1}{v+1} = 0$

$\int \frac{v+1}{v^2+1}\,dv + \int \frac{dx}{x} = c$

Let $v = \tan\theta \quad \therefore dv = \sec^2\theta\, d\theta$

$\int \frac{\tan\theta + 1}{\sec^2\theta} \sec^2\theta\, d\theta + \log x = c$

$\log\sec\theta + \theta + \log x = c$

$\left\{ \because \tan\theta = \frac{v}{1} \;\therefore \cos\theta = \frac{1}{\sqrt{v^2+1}} \;\therefore \sec\theta = \sqrt{v^2+1} \right\}$

$\log\left(\sqrt{v^2+1}\right) + \tan^{-1} v + \log x = c$

$\log\sqrt{\frac{y^2+x^2}{x^2}} + \tan^{-1}\left(\frac{y}{x}\right) + \log x = c$

$= \log\sqrt{x^2+y^2} - \cancel{\log\sqrt{x^2}} + \tan^{-1}(y/x) + \cancel{\log x} = c$

Tel: 9820246760 / 9769479368 / 9820009639

Tel: 9820246760 / 9769479368 / 9820009639

$\therefore \log\sqrt{x^2+y^2} + \tan^{-1}\left(\frac{y}{x}\right) = c$

5

$y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$y^2 = (xy - x^2)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{y^2}{xy - x^2}$

Let $y = vx \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore v + x\frac{dv}{dx} = \frac{v^2x^2}{x\,vx - x^2} = \frac{v^2}{v-1}$

$x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v^2 + v}{v-1}$

$\int \frac{v-1}{v}\,dv - \int \frac{dx}{x} = c$

$\int\left(1 - \frac{1}{v}\right)dv - \int\frac{dx}{x} = c$

$v - \log v - \log x = c$

$\log v + \log x - v = c$

$\log vx - v = \log c'$

$\log y - \frac{y}{x} = \log c'$

$\log y - \log c' = y/x$

$\log\frac{y}{c'} = \frac{y}{x}$

$\frac{y}{c'} = e^{y/x}$

$y = c'e^{y/x}$

Eg 1: $(3y-7x+7)dx+(7y-3x+3)dy=0$

Since it is an non-homogeneous linear eq.

$$\therefore \frac{dy}{dx} = -\frac{(3y-7x+7)}{7y-3x+3}$$

$$= \frac{7x-3y-7}{7y-3x+3} = \frac{7X-3Y+7h-3k-7}{7Y-3X-3h+7k+3}$$

Let $x = X+h$; $y = Y+k$

$7h-3k-7=0 \quad \times 3$

$-3h+7k+3=0 \quad \times 7$

$21h-9k-21=0$

$-21h+49k+21=0$

$40k = 0 \quad \therefore k=0$

$7h=7 \quad \therefore h=1$

$$\frac{dY}{dX} = \frac{7X-3Y}{7Y-3X}$$

Is an homogeneous eq.

Let $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$v + X\frac{dv}{dX} = \frac{7X-3vX}{7vX-3X}$$

$$\therefore X\frac{dv}{dX} = \frac{7-3v}{7v-3} - v$$

$$\therefore X\frac{dv}{dX} = \frac{7-3v-7v^2+3v}{7v-3} = \frac{7-7v^2}{7v-3}$$

$$\frac{1}{7}\int \frac{7v-3}{1-v^2}\,dv - \int \frac{dX}{X} = c$$

$$\frac{1}{7}\left[\frac{7}{-2}\int \frac{-2v\,dv}{(1-v^2)} - 3\int \frac{dv}{1-v^2}\right] - \int \frac{dX}{X} = c$$

$$\frac{1}{7}\left[\frac{7}{-2}\log((1-v)(1+v)) - \frac{3}{2}\log\left(\frac{1+v}{1-v}\right)\right] - \log X = c$$

$$\frac{1}{7}\left[-\frac{7}{2}\log(1-v) - \frac{7}{2}\log(1+v) - \frac{3}{2}\log(1+v) + \frac{3}{2}\log(1-v)\right] - \log X = c$$

$+2\log(1-v)+5\log(1+v)+7\log x=\log c$

$\log (1-v)^2(1+v)^5 \cdot x^7=\log c$

$\left(1-\frac{y}{x}\right)^2\left(1+\frac{y}{x}\right)^5 \cdot x^7=c$

$(x-y)^2(x+y)^5=c$

$\therefore X=x-1;\ Y=y$

$(x-1-y)^2(x-1+y)^5=c$

Q2

$(6x-4y+1)\frac{dy}{dx}=3x-2y+1$

$\frac{dy}{dx}=\frac{3x-2y+1}{2(3x-2y)+1}$

Let $3x-2y=v$

$\therefore 3-2\frac{dy}{dx}=\frac{dv}{dx}$

$\frac{dy}{dx}=\frac{1}{2}\left(3-\frac{dv}{dx}\right)$

$\frac{1}{2}\left(3-\frac{dv}{dx}\right)=\frac{v+1}{2v+1}$

$3-\frac{dv}{dx}=\frac{2v+2}{2v+1}$

$\frac{dv}{dx}=3-\frac{2v+2}{2v+1}=\frac{6v+3-2v-2}{2v+1}$

$\frac{dv}{dx}=\frac{4v+1}{2v+1}$

$\int\frac{2v+1}{4v+1}dv-\int dx=c$

$\frac{1}{2}\int\frac{4v+1+1}{4v+1}dv-\int dx=c$

$\frac{1}{2}\left[\int dv+\int\frac{dv}{4v+1}\right]-\int dx=c$

$\frac{1}{2}\left[v+\frac{1}{4}\log(4v+1)\right]-x=c$

$\frac{1}{2}v+\frac{1}{8}\log(4v+1)-x=c$

$4v + \log(4v+1) + 8x = c$

$4(3x-2y) + \log(12x-8y+1) - 8x = c$

$4x - 8y + \log(12x-8y+1) = c$

Eq. II-C

long 1

$(y-2x)dx + (2y-3x+1)dy = 0$

$\frac{dy}{dx} = \frac{2x-y}{2y-3x+1}$

∴ Eq is non-homogeneous Linear eq

Let $x = X+h$, $y = Y+k$

$\frac{dY}{dX} = \frac{2X-Y+2h-k}{2Y-3X+2k-3h+1}$

$2h-k=0 \quad \times 2$

$+3h-2k=1 \quad \times 1$

$4h-2k=0$

$-3h \mp 2k = -1$

$h=-1 \quad \therefore k=-2$

$X=x+1$
$Y=y+2$

$\frac{dY}{dX} = \frac{2X-Y}{2Y-3X}$

∴ Eq is homogeneous

Let $Y = vX$

$\therefore \frac{dY}{dX} = v + X\frac{dv}{dX}$

$v + X\frac{dv}{dX} = \frac{2X-vX}{2vX-3X}$

$X\frac{dv}{dX} = \frac{2-v}{2v-3} - v = \frac{2-v-2v^2+3v}{2v-3}$

$\int \frac{2v-3}{-2v^2+2v+2}dv - \int \frac{dX}{X} = c \quad (\times(-1))$

$\frac{1}{2}\int \frac{2v-3}{v^2-v-1}dv + \int \frac{dX}{X} = c$

$\cancel{\frac{2}{2}\int \frac{2v+2-5}{v^2+2v+1}dv + \log X = c}$

$\cancel{\int dv - \int \frac{\lambda\, dv}{(v+1)^2} + \log X = c}$

Tel: 9820246760 / 9769479368 / 9820009639

HW 2

$$\frac{dy}{dx} = \frac{-2x+3y}{y+2}$$

$\therefore$ It is non homogeneous eq.

Let $x = X+h$, $y = Y+k$

$$\frac{dY}{dX} = \frac{-2X+3Y-2h-3k}{Y+k+2}$$

$-2h-3k=0$

$k=-2 \quad \therefore h=3$

$$\frac{dY}{dX} = \frac{-2X-3Y}{Y}$$ is homogeneous eq.

Let $Y = vX$

$$\therefore \frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$\therefore v + X\frac{dv}{dX} = \frac{-2X-3vX}{vX}$$

$$X\frac{dv}{dX} = \frac{-2-3v}{v} - v = \frac{-2-3v-v^2}{v}$$

$$\frac{1}{2}\int\frac{2v\,dv}{v^2+3v+2} + \int\frac{dX}{X} = c$$

$$\frac{1}{2}\int\frac{2v+3-3}{v^2+3v+2}dv + \log X = c$$

$$\frac{1}{2}\int\frac{2v+3\,dv}{v^2+3v+2} - \frac{3}{2}\int\frac{dv}{v^2+3v+\frac{9}{4}-\frac{1}{4}} + \log X = c$$

$$\frac{1}{2}\log(v^2+3v+2) - \frac{3}{2}\int\frac{dv}{\left(v+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2} + \log X = c$$

$$\frac{1}{2}\log(v^2+3v+2) - \frac{3}{2}\cdot\frac{2}{2}\log\left(\frac{v+\frac{3}{2}-\frac{1}{2}}{v+\frac{3}{2}+\frac{1}{2}}\right) + \log X = c$$

$$\log\sqrt{v^2+3v+2} - \log\left(\frac{v+1}{v+2}\right)^{3/2} + \log X = \log c$$

$$\log\sqrt{v^2+3v+2}\cdot\left(\frac{v+2}{v+1}\right)^{3/2}\cdot X = \log c$$

$$\sqrt{\frac{Y^2}{X^2}+\frac{3Y}{X}+2}\cdot\left(\frac{Y+2X}{Y+X}\right)^{3/2}\cdot X = c$$

$$\frac{\sqrt{Y^2+3XY+2X^2}}{\cancel{X}}\cdot\left(\frac{Y+2X}{Y+X}\right)^{3/2}\cdot\cancel{X} = c$$

$y = Y + 2$, $x = X - 3$

$$\cancel{\left[(y+2)^2 + 3(x-3)(y+2) + 2(x-3)^2\right] \cdot \left(\frac{y+2+2x-6}{y+2+x-3}\right)^3 = c}$$

$$\cancel{\left[y^2 + 2y + 4 + 3xy + 6x - 9y - 18 +\right.}$$

$$\sqrt{Y^2 + XY + 2XY + 2X^2} \cdot \left(\frac{Y+2X}{Y+X}\right)^{3/2} = c$$

$$\sqrt{Y(Y+X) + 2X(Y+X)} \cdot \left(\frac{Y+2X}{Y+X}\right)^{3/2} = c$$

$$\sqrt{(Y+2X)(Y+X)} \cdot \left(\frac{Y+2X}{Y+X}\right)^{3/2} = c$$

$$\frac{(Y+2X)^{3/2+1/2}}{(Y+X)^{3/2-1/2}} = C$$

$$\frac{(Y+2X)^2}{(Y+X)} = C$$

$$(Y+2X)^2 = C(Y+X)$$

$$(y+2+2x-6)^2 = C(y+2+x-3)$$

$$\therefore (2x+y-4)^2 = C(x+y-1)$$

HW 3 $(2x-y+1)dx + (2y-x-1)dy = 0$

$$\frac{dy}{dx} = \frac{2x-y+1}{x-2y+1}$$

$\therefore$ non-homogeneous eq

$x = X + h$; $y = Y + k$

$$\frac{dY}{dX} = \frac{2X - Y + 2h - k + 1}{X - 2Y + h - 2k + 1}$$

$$2h - k + 1 = 0$$
$$- \; 2h - 4k + 2 = 0$$
$$3k - 1 = 0 \quad ; k = 1/3 ; h = -1/3$$

$$\frac{dY}{dX} = \frac{2X - Y}{X - 2Y}$$

Is homogeneous eq.

Tel: 9820246760 / 9769479368 / 9820009639

Let $y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{2x - vx}{x + 2vx}$

$x\frac{dv}{dx} = \frac{2-v}{1-2v} - v = \frac{2 - v - v + 2v^2}{1-2v}$

$x\frac{dv}{dx} = \frac{2 - 2v + 2v^2}{1-2v}$

$\frac{1}{2}\int \frac{2v-1}{v^2-v+1}\,dv + \int \frac{dx}{x} = c$

$\frac{1}{2}\log(v^2 - v + 1) + \log x = c$

$\log \sqrt{v^2 - v + 1} \times x = \log c'$

$\left(\frac{y^2}{x^2} - \frac{y}{x} + 1\right)x^2 = c'$

$y^2 - xy + x^2 = c'$

$y = Y - \frac{1}{3}$, $x = X + \frac{1}{3}$

$\left(y - \frac{1}{3}\right)^2 - \left(x + \frac{1}{3}\right)\left(y - \frac{1}{3}\right) + \left(x + \frac{1}{3}\right)^2 = c'$

$\frac{(3y-1)^2}{9} - \frac{(3x+1)(3y-1)}{9} + \frac{(3x+1)^2}{9} = \frac{c'}{9}$

$9y^2 - 6y + 1 - 9xy + 1 + 3x - 3y + 9x^2 + 6x + 1$

$= 9c'$

$9x^2 - 9xy + 9y^2 + 9x - 9y = c'' 9 - 3$

$\therefore x^2 - xy + y^2 + x - y = c''' \quad \text{where } c''' = \frac{c'' - 3}{9}$

HW

4) $(6x + 9y + 6)\frac{dy}{dx} = 2x + 3y - 1$

$\frac{dy}{dx} = \frac{2x + 3y - 1}{6x + 9y + 6}$ is a non-homogeneous eq

Let $X = x + h$, $Y = y + k$

$\frac{dY}{dX} = \frac{2X + 3Y + 2h + 3k - 1}{6X + 9Y + 6h + 9k + 6}$

Tel: 9820246760 / 9769479368 / 9820009639

navlakhi.com navlakhi.mobi navlakhi.tv

$\cancel{2x+3y=1}$
$\cancel{6x+9y=-6}$

$$\frac{dy}{dx} = \frac{2x+3y-1}{3(2x+3y)+6}$$

Let $2x+3y = v$

$\therefore 2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore \frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$$

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{v-1}{3(v+2)}$$

$$\therefore \frac{dv}{dx} = \frac{v-1}{v+2} + 2 = \frac{v-1+2v+4}{v+2}$$

$$\frac{dv}{dx} = \frac{3v+3}{v+2} = \frac{3(v+1)}{v+2}$$

$$\frac{1}{3}\int \frac{3v-6}{3v-5}\,dv - \int dx = c$$

$$\frac{1}{3}\left[\int \frac{3v-5}{3v-5}dv - \int \frac{dv}{3v-5}\right] - x = c$$

$$\int dv - \int \frac{dv}{3v-5} - 3x = 3c$$

$$v - \frac{1}{3}\log(3v-5) - 3x = c'$$ where $c' = 3c$

$$3v - \log(3v-5) - 9x = c''$$ " $c'' = 3c'$

$$3(2x+3y) - \log[3(2x+3y)-5] - 9x = c''$$

$$6x+9y - \log(6x+9y-5) - 9x = c''$$

$$-3x+9y-\log(6x+9y-5) = c''$$

$$\log(6x+9y-5) + \log c''' = 9y - 3x$$

$$\frac{1}{3}\int \frac{v+2}{v+1}\,dv - \int dx = c$$

$$\frac{1}{3}\left[\int \frac{v+1}{v+1}dv + \int \frac{dv}{v+1}\right] - x = c$$

$$\int dv + \int \frac{dv}{v+1} - 3x = c' \qquad (c' = 3c)$$

$$v + \log(v+1) - 3x = c'$$

$$2x + 3y + \log(2x+3y+1) - 3x = c'$$

$$3y - x + \log(2x+3y+1) = c'$$

$$\log(2x+3y+1) - \log c'' = x - 3y$$

$$\log\left(\frac{2x+3y+1}{c''}\right) = x - 3y$$

$$2x+3y+1 = c'' e^{(x-3y)}$$

[DO METHOD II BEFORE THIS]

Method V: I.F. (Integrating factor) method to make the given eq. exact

I.F. is a term multiplied throughout the D.E. in order to make the eq. exact

navlakhi.com navlakhi.mobi navlakhi.tv

Tel: 9820246760 / 9769479368 / 9820009639

Eg 1 $y\sec^2 x\,dx + \left[3\tan x - \frac{\sec^2 y}{y^2}\right]dy = 0$

has I.F. y^n, find n. hence solve eq.

$M\,dx + N\,dy = 0$

$\because$ IF is y^n

$\therefore y^{n+1}\sec^2 x\,dx + y^n\left[3\tan x - \frac{\sec^2 y}{y^2}\right]dy = 0$ — (I)

After multiplying the IF eq. becomes ~~homogeneous~~ exact

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

II-5hr T-5 $\therefore y^n(n+1)\sec^2 x = 3y^n\sec^2 x$

$\therefore n+1 = 3 \quad \therefore n = 2$

Sub. $n=2$ in (I)

I-3hr T-5 $y^3\sec^2 x\,dx + (3y^2\tan x - \sec^2 y)dy = 0$

Is an ~~homogeneous~~ eq. $\therefore$ Soln. is

$\int_{y\text{ const}} y^3\sec^2 x\,dx + \int_{\text{only }y\text{ terms}} -\sec^2 y\,dy = c$

$y^3\tan x - \tan y = c$

(a) I.F by inspection — X

(b) I.F. for homogeneous equation $= \frac{1}{Mx+Ny}$

Eg 1 $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Since it is an homogeneous eq.

$\therefore$ I.F. $= \frac{1}{Mx+Ny} = \frac{1}{(x^2y-2xy^2)x - (x^3-3x^2y)y}$

$= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$

~~Eq. becomes~~

$\frac{1}{x^2y^2}(x^2y - 2xy^2)dx - \frac{1}{x^2y^2}(x^3 - 3x^2y)dy = 0$

$\therefore \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$

Tel: 9820246760 / 9769479368 / 9820009639

$\therefore$ Eq. is exact $\therefore$ soln. is

$$\int_{y\ \text{const}} M\,dx + \int_{\text{no } x \text{ terms}} N\,dy = c$$

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = c$$

$$\therefore \frac{x}{y} - 2\log x + 3\log y = \log c'$$

$$\log \frac{y^3}{2x^2c'} = -\frac{x}{y}$$

$$\therefore y^3 = c''x^2 e^{-x/y}$$

(b)(c) I.f. for eq. of type $y f_1(xy)\,dx + x f_2(xy)\,dy = 0$ is $\dfrac{1}{Mx - Ny}$

Eg: $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$ is of the form –

$$y f_1(xy)\,dx + x f_2(xy)\,dy = 0$$

$$\therefore \text{I.f.} = \frac{1}{Mx - Ny}$$

$$= \frac{1}{xy(xy + 2x^2y^2) - xy(xy - x^2y^2)}$$

$$= \frac{1}{xy(\cancel{xy} + 2x^2y^2 - \cancel{xy} + x^2y^2)}$$

$$= \frac{1}{3x^3y^3}$$

$$\therefore \frac{1}{3x^3y^2}(xy + 2x^2y^2)\,dx + \frac{1}{3x^2y^3}(xy - x^2y^2)\,dy = 0$$

$$\therefore \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

$\therefore$ Eq. is exact $\therefore$ soln. is

$$\int_{y\ \text{const}} M\,dx + \int_{\text{no } x \text{ terms}} N\,dy = C$$

Tel: 9820246760 / 9769479368 / 9820009639

navlakhi.com navlakhi.mobi navlakhi.tv

$$\int\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\int -\frac{dy}{y}=c$$

$$\therefore -\frac{1}{xy}+2\log x-\log y=c$$

$$\log\frac{x^2}{yC}=\frac{1}{xy}$$

$$x^2=cye^{1/xy}$$

(d) For an eq. of type $Mdx+Ndy=0$

if $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ is a funct. of x alone say $f(x)$

then I.F. $=e^{\int f(x)dx}$

& if $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$ is a funct. of y alone say $f(y)$

then I.F. $=e^{\int f(y)dy}$

Eg1 $(x^2+y^2+2x)dx+2y\,dy=0$

is of the form

$$Mdx+Ndy=0$$

$$\therefore M=x^2+y^2+2x;\ N=2y$$

$$\therefore \frac{\partial M}{\partial y}=2y;\ \frac{\partial N}{\partial x}=0$$

$$\therefore \frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{2y-0}{2y}=1=f(x)$$

$$\therefore I.F.=e^{\int 1dx}=e^x$$

$$\therefore e^x(x^2+y^2+2x)dx+2e^xy\,dy=0$$

$\therefore$ Eq. is exact $\therefore$ its soln. is

$$\int_{y\text{ const}} M\,dx+\int_{\text{no }x\text{ terms}} N\,dy=c$$

$$\int(e^xx^2+e^xy^2+2e^xx)\,dx+0=c$$

$$\int e^xx^2+e^xy^2+2\int xe^x=c$$

$x^2\int e^x dx - \int 2x e^x dx + y^2 e^x + 2\int x e^x = c$

$\therefore x e^x + y^2 e^x = c$

$\therefore e^x(x+y^2) = c$

Eg 2 $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

is of the form

$M\,dx + N\,dy = 0$

$\therefore M = 3x^2y^4 + 2xy;\ N = 2x^3y^3 - x^2$

$\therefore \frac{\partial M}{\partial y} = 12x^2y^3 + 2x;\ \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$

$\therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-6x^2y^3 - 4x}{3x^2y^4 + 2xy}$

$= \frac{-2}{y}\left(\frac{3x^2y^3 + 2x}{3x^2y^3 + 2x}\right) = \frac{-2}{y}$

$\therefore IF = e^{\int -2/y\, dy} = e^{-2\log y} = e^{\log y^{-2}}$

$= y^{-2} = \frac{1}{y^2}$

$\therefore \frac{1}{y^2}(3x^2y^4 + 2xy)dx + \frac{1}{y^2}(2x^3y^3 - x^2)dy = 0$

$\therefore \left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0$

$\therefore$ Eq. is exact $\therefore$ soln is

$\int_{y\ \text{const}} M\,dx + \int_{\text{no } x \text{ terms}} N\,dy = c$

$\int \left(3x^2y^2 + \frac{2x}{y}\right)dx + 0 = c$

$\therefore 3y^2\frac{x^3}{3} + \frac{2}{y}\frac{x^2}{2} = c$

$\therefore x^3y^2 + \frac{x^2}{y} = c$

$\therefore x^3y^3 + x^2 = cy$

last

(e) d For an eq. of the type:-

$$x^{\alpha} y^{\beta}(my\,dx + nx\,dy) + x^{\alpha'} y^{\beta'}(m'y\,dx + n'x\,dy) = 0$$

I.F. is $x^{km-1-\alpha}\, y^{kn-1-\beta}$

or $x^{k'm'-1-\alpha'}\, y^{k'n'-1-\beta'}$

Solve,

$$km-1-\alpha = k'm'-1-\alpha'$$

& $$kn-1-\beta = k'n'-1-\beta'$$

to get values of k & k'

Eg 1 $(2x^2y^2+y)\,dx - (x^3y-3x)\,dy = 0$

$$y(2x^2y+1)\,dx - x(x^2y-3)\,dy = 0$$

$$(2x^2y \cdot y\,dx - x^2y \cdot x\,dy) + (1 \cdot y\,dx + 3 \cdot x\,dy) = 0$$

$$x^2y(2y\,dx - x\,dy) + 1(y\,dx + 3x\,dy) = 0$$

is of the form:-

$$x^{\alpha} y^{\beta}(my\,dx + nx\,dy) + x^{\alpha'} y^{\beta'}(m'y\,dx + n'x\,dy) = 0$$

$\therefore$ IF $= x^{km-1-\alpha} \cdot y^{kn-1-\beta}$

$\alpha = 2,\ \beta = 1,\ m = 2,\ n = -1,$

$\alpha' = 0,\ \beta' = 0,\ m' = 1,\ n' = 3$

Also,

$\cancel{km-1-\alpha = kn-1-\beta}$

& $\cancel{k'm'-1-\alpha' = k'n'-1-\beta'}$

$\therefore \cancel{2k-1-2 = -k-1-1}$

$\cancel{k'-1-0 = 3k'-1-0}$

$$km-1-\alpha = k'm'-1-\alpha'$$

$$kn-1-\beta = k'n'-1-\beta'$$

$\therefore 2k-1-2 = k'-1-0$

$-k-1-1 = 3k'-1-0$

$\therefore 2k-k' = 2 \quad \times 1$

$k+3k' = -1 \quad \times 2$

$$2k - k' = 2$$
$$-2k + 6k' = -2$$
$$\cancel{7}k' = 4 \quad \therefore k' = 4/\cancel{7}$$

$$-k = -1 - 3k'$$
$$k = -1 + \frac{12}{\cancel{7}} = \frac{5}{7}$$

$$I.F. = x^{k'm'-1-\alpha'}\, y^{k'n'-1-\beta'}$$

$$I.F. = x^{-\frac{4}{7}-1}\, y^{-\frac{12}{7}-1} = x^{-11/7}\, y^{-19/7}$$

$$\therefore x^{-11/7} y^{-19/7}(2x^2y^2 + y)dx$$
$$- x^{-11/7} y^{-19/7}(x^3y - 3x)dy = 0$$

$$\left(2x^{3/7}y^{-5/7} + x^{-11/7}y^{-12/7}\right)dx$$
$$- x^{-4/7}y^{-19/7}(x^3y - 3x)dy = 0$$

$\therefore$ Eq. is exact $\therefore$ Soln. is

$$\int M dx + \int N dy = c$$

y const   no x terms

$$\int\left(2x^{3/7}y^{-5/7} + x^{-11/7}y^{-12/7}\right)dx + 0 = c$$

$$2\frac{x^{10/7}y^{-5/7}}{10_5}\times 7 + \frac{x^{-4/7}y^{-12/7}}{-4}\times 7 = \frac{c}{7}$$

$$\therefore \frac{x^{10/7}y^{-5/7}}{5} - \frac{x^{-4/7}y^{-12/7}}{4} = c' \quad \left(c' = \frac{c}{7}\right)$$

Eg. II – E

1. $y\,dx - x\,dy + \log x\,dx = 0$

$$(y + \log x)dx - x\,dy = 0$$

is of the form

$$M dx + N dy = 0$$

$M = y + \log x \; ; \; N = -x$

$\frac{\partial M}{\partial y} = 1 \; ; \; \frac{\partial N}{\partial x} = -1$

~~$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ Eq. is exact~~

~~$\therefore$ Soln is~~

~~$\int M\,dx + \int N\,dy = c$~~

~~y const no x terms~~

~~$\int (y + \log x)\,dx + 0 = c$~~

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1+1}{-x} = -\frac{2}{x} = f(x)$ alone

$\therefore$ I.F. $= e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx}$

$= e^{-2\log x} = e^{\log x^{-2}} = \frac{1}{x^2}$

$\therefore \frac{1}{x^2}(y + \log x)dx - \frac{x}{x^2}dy = 0$

$\left(\frac{y}{x^2} + \frac{\log x}{x^2}\right)dx - \frac{1}{x}dy = 0$

$\therefore$ Eq. is exact $\therefore$ G.S. is

$\int M\,dx + \int N\,dy = c$

y const no x terms

$\int \left(\frac{y}{x^2} + \log x \frac{1}{x^2}\right)dx + 0 = c$

$-\frac{y}{x} + \log x\left(-\frac{1}{x}\right) - \int \frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx = c$

$-\frac{y}{x} - \frac{\log x}{x} + \int \frac{1}{x^2}dx = c$

$-\frac{y}{x} - \frac{\log x}{x} - \frac{1}{x} = c$

$\therefore y + \log x + 1 = cx$

$-cx + y + \log x + 1 = 0$

$\therefore c'x + y + \log x + 1 = 0$

$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Eq. is of the form:-

$y f_1(xy)dx + x f_2(xy)dy = 0$

$IF = \frac{1}{Mx - Ny}$

$= \frac{1}{xy(xy+2x^2y^2) - xy(xy - x^2y^2)}$

$= \frac{1}{xy(xy - 2x^2y^2 - xy + x^2y^2)}$

$= \frac{+1}{3x^3y^3}$

$\frac{1}{x^3y^2}(xy + 2x^2y^2)dx + \frac{1}{x^2y^3}(xy - x^2y^2)dy = 0$

$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$

∴ Eq. is exact ∴ G.S. is

$\int M dx + \int N dy = c$

y const   no x terms

$\int\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx - \int\frac{dy}{y} = c$

$-\frac{1}{xy} + 2\log x - \log y = c$

$\log\frac{x^2}{cy} = \frac{1}{xy}$

$x^2 = cye^{1/xy}$

x Homog

3. $(x^2+y^2)dx - 2xy\,dy = 0$

∴ Eq is homogeneous

Let $y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2x.vx} = \frac{1+v^2}{2v}$

$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$

$$\therefore \int \frac{2v}{v^2-1}\,dv + \int \frac{dx}{x} = c$$

$$\log(v^2-1) + \log x = \log c' \quad (c = \log c')$$

$$\log(v^2-1)\cdot x = \log c'$$

$$\left(\frac{y^2}{x^2}-1\right)\cdot x = c'$$

$$y^2 - x^2 = c'x$$

$$x^2 - y^2 = -c'x$$

$$x^2 - y^2 = c''x$$

HW $\times$ Homog

4. $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$

$\therefore$ Eq is homogeneous

$$\therefore \frac{dy}{dx} = \frac{y^3 - 2yx^2}{x^3 - 2xy^2}$$

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore v + x\frac{dv}{dx} = \frac{v^3x^3 - 2vx\cdot x^2}{x^3 - 2x\cdot v^2x^2}$$

$$x\frac{dv}{dx} = \frac{v^3 - 2v}{1-2v^2} - v$$

$$= \frac{v^3 - 2v - v + 2v^3}{1-2v^2} = \frac{3v^3 - 3v}{1-2v^2}$$

$$\frac{1}{3}\int \frac{2v^2-1}{v^3-v}\,dv + \int \frac{dx}{x} = c$$

$$\cancel{\frac{1}{3}\log(v^3-v) + 3\log x = 3\log c'}$$

$$\cancel{\frac{1}{3}\log(v^3-v)\cdot x^3 = \log(c')^3}$$

$$\cancel{\left(\frac{y^3}{x^3} - \frac{y}{x}\right)\cdot x^3 = c''}$$

$$\cancel{y^3 - x^2y = c''}$$

$$\frac{1}{3}\int \frac{2v^2 - 1 + \boxed{v^2 - v^2}}{v^3 - v}\,dv + \log x = \log c'$$

$\int \frac{3v^2-1}{v^3-v}\,dv - \int \frac{v^2\,dv}{v^3-v} + 3\log x = 3\log c'$

$\log(v^3-v) - \frac{1}{2}\int \frac{2v^2\,dv}{v(v^2-1)} + \log x^3 = \log c'$

$2\log(v^3-v) - \frac{1}{2}\log(v^2-1) + 2\log x^3 = 2\log c''$

$\log \frac{(v^3-v)^2}{v^2-1} \times x^{3\times 2} = \log c'''$

$\frac{v^2(v^2-1)^2}{(v^2-1)} \cdot x^6 = c'''$

$\frac{y^2}{x^2}\left(\frac{y^2}{x^2}-1\right) \cdot x^6 = c'''$

$\frac{y^2}{x^4}(y^2-x^2) \cdot x^6 = c'''$

$x^2y^2(y^2-x^2) = c'''$

S $(x^2+y^2+1)\,dx - 2xy\,dy = 0$

Eq. is of the form:-

$M\,dx + N\,dy = 0$

$M = x^2+y^2+1 \quad ; \quad N = -2xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = -2y$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y+2y}{-2xy} = \frac{4y}{-2xy} = \frac{-2}{x} = f(x)$

$I.F. = e^{\int f(x)\,dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}}$

$= 1/x^2$

$\frac{1}{x^2}(x^2+y^2+1)\,dx + \frac{1}{x^2}(-2xy)\,dy = 0$

$\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx - \frac{2y}{x}\,dy = 0$

Is an homogeneous eq. ∴ G.S. is

$\int M\,dx + \int N\,dy = C$

y const. no x terms

$$x - \frac{y^2}{x} - \frac{1}{x} = c$$

$$x^2 - y^2 - 1 = cx$$

6 $y(xy+1)dx + x(1+xy+x^2y^2)dy = 0$

Is of the form

$y f_1(xy)dx + x f_2(xy)dy = 0$

$$I.F. = \frac{1}{Mx - Ny}$$

$$= \frac{1}{xy(xy+1) - xy(1+xy+x^2y^2)}$$

$$= \frac{1}{xy(\cancel{xy}+\cancel{1}-1-xy-x^2y^2)}$$

$$= \frac{-1}{x^3y^3}$$

$$\frac{1}{x^3y^2}(xy+1)dx + \frac{1}{x^2y^3}(1+xy+x^2y^2)dy = 0$$

$$\left(\frac{1}{x^2y} + \frac{1}{x^3y^2}\right)dx + \left(\frac{1}{x^2y^3} + \frac{1}{xy^2} + \frac{1}{y}\right)dy = 0$$

$\therefore$ Eq. is exact $\therefore$ GS is

$$\int\limits_{y\ \text{const.}} M dx + \int\limits_{\text{no } x \text{ terms}} N dy = c$$

$$\int\left(\frac{1}{x^2y} + \frac{1}{x^3y^2}\right)dx + \int\frac{dy}{y} = c$$

$$\therefore -\frac{1}{xy} - \frac{1}{2x^2y^2} + \log y = c$$

$$2x^2y^2\log y - 2xy - 1 = 2c\,x^2y^2$$

$$= c'x^2y^2$$

~~7 $ye^x dx = (y^3 + 2xe^y)dy$~~

~~$ye^x dx - (y^3 + 2xe^y)dy = 0$~~

~~Is of the form~~

~~$M dx + N dy = 0$~~

~~$M = ye^x$; $N = -y^3 - 2xe^y$~~

~~$\frac{\partial M}{\partial y} = e^x$; $\frac{\partial N}{\partial x} = -2e^y$~~

7 $ye^{-y}dx - (y^3 + 2xe^{-y})dy = 0$

$$\frac{\partial M}{\partial y} = e^{-y} - ye^{-y}$$

$$\frac{\partial N}{\partial x} = -2e^{-y}$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{ye^{-y} - 3e^{-y}}{ye^{-y}}$$

$$= \frac{y-3}{y} = f(y)$$

$$I.F. = e^{\int f(y)dy} = e^{\int (1-\frac{3}{y})dy}$$

$$= e^{y-3\log y} = \frac{e^y}{e^{3\log y}} = \frac{e^y}{y^3}$$

$$\frac{e^y}{y^3} \cdot ye^{-y}dx - \frac{e^y}{y^3}(y^3 + 2xe^{-y})dy = 0$$

$\therefore$ Eq is exact $\therefore$ GS is

$$\frac{x}{y^2} - e^y = C \Rightarrow -\frac{x}{y^2} + e^y = c'$$

METHOD VI : Linear Equations:

(a) Linear Eq. of first order –

Eq. of the form

$$\frac{dy}{dx} + Py = Q$$

where P, Q are functions of x only or constants,

have the solution

$$y(I.F.) = C + \int Q(I.F.)\,dx$$

where $I.F. = e^{\int P dx}$

Eg 1 $\frac{dy}{dx} - 2y = e^{2x}$

is of the form

$\frac{dy}{dx} + Py = Q$

$P = -2$; $Q = e^{2x}$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int -2 dx} = e^{-2x}$

G.S. is

$y(\text{I.F.}) = c + \int Q(\text{I.F.})dx$

$\therefore ye^{-2x} = c + \int \cancel{e^{2x}} \cdot \cancel{e^{-2x}} dx$

$\therefore ye^{-2x} = c + x$

$\therefore y = e^{2x}(c+x)$

Eg 2 $x(1-x^2)\frac{dy}{dx} + (2x^2-1)y = x^3$

$\div$ by $x(1-x^2)$

$$\therefore \frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}\, y = \frac{x^3}{x(1-x^2)}$$

is of the form:-

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{2x^2-1}{x(1-x^2)} \; ; \; Q = \frac{x^3}{x(1-x^2)}$$

$$P = \frac{2x^2-1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

For A: put $x=0 \therefore A = -1$

For B: put $x=1 \therefore B = 1/2$

For C: put $x=-1 \therefore C = -1/2$

$$\therefore P = \frac{-1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}.$$

$$I.F. = e^{\int P\,dx}$$

$$\int P\,dx = -\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(x+1)$$

$$= -\left(\log x + \log(1-x)^{1/2} + \log(1+x)^{1/2}\right)$$

$$= \left(\log \frac{1}{x(1-x)^{1/2}(1+x)^{1/2}}\right)$$

$$I.F. = e^{\int P\,dx} = e^{\log \frac{1}{x(1-x)^{1/2}(1+x)^{1/2}}}$$

$$= \frac{1}{x(1-x)^{1/2}(1+x)^{1/2}} = \frac{1}{x\sqrt{1-x^2}}$$

G.S. is

$$y(I.F.) = c + \int Q(I.F.)\,dx$$

$$\frac{y}{x\sqrt{1-x^2}} = c + \int \frac{x^2}{(1-x^2)} \times \frac{1}{x(1-x^2)^{1/2}}\,dx$$

$$= c - \frac{1}{2}\int \frac{-2x\,dx}{(1-x^2)^{3/2}}$$

$$= c - \frac{1}{2} \times \frac{(1-x^2)^{-1/2}}{-1/2}$$

$$\frac{y}{x\sqrt{1-x^2}} = c + \frac{1}{\sqrt{1-x^2}}$$

$$\therefore y = cx\sqrt{1-x^2} + x$$

Eq 2(b)(i) Eq. reducible to linear form:
Bernoulli's Eq. of the form
$\frac{dy}{dx} + Py = Qy^n$ where P & Q are funct. of x only or const

can be made linear by ÷ by y^n

$\therefore y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$

Let $y^{1-n} = z$; diff.

$\therefore (1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$

Sub.

$\therefore \frac{1}{1-n}\frac{dz}{dx} + Pz = Q$

$\therefore \frac{dz}{dx} + (1-n)Pz = Q(1-n)$ is an linear eq. with z as dependent variable.

Eg 3 $\frac{dy}{dx} = x^3y^3 - xy$ (of the form $\frac{dy}{dx} + Py = Qy^n$)

$\frac{dy}{dx} + xy = x^3y^3$ (÷ by y^3)

$y^{-3}\frac{dy}{dx} + xy^{-2} = x^3$

Let $y^{-2} = z$ $\therefore -2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}$

$\therefore -\frac{1}{2}\frac{dz}{dx} + xz = x^3$

$\therefore \frac{dz}{dx} - 2xz = -2x^3$

is of the form

$\frac{dz}{dx} + Pz = Q$

$P = -2x$; $Q = -2x^3$

$$IF = e^{\int P dx} = e^{\int -2x dx} = e^{-x^2}$$

$\therefore$ a.s. is

$$z(IF) = c + \int Q(IF)\,dx$$

$$ze^{-x^2} = c + \int -2x^3 e^{-x^2} dx$$

$$ze^{-x^2} = c + \int x^2 e^{-x^2} dx(-2x)$$

Put $-x^2 = t$ $\therefore -2x\,dx = dt$

$$\therefore ze^{-x^2} = c - \int t e^t dt$$

$$ze^{-x^2} = c - te^t + e^t$$

$$ze^{-x^2} = c + x^2 e^{-x^2} + e^{-x^2}$$

$$\therefore z = ce^{x^2} + x^2 + 1$$

$$y^{-2} = ce^{x^2} + x^2 + 1$$

$\frac{I-3hr}{T=8}$

b(ii) Eq. of the type :-

$$f'(y)\frac{dy}{dx} + Pf(y) = Q$$

where P & Q are funct. of x only or const.

Let $z = f(y)$

$$\therefore \frac{dz}{dx} = f'(y)\frac{dy}{dx}$$

Subs. $\therefore$ Eq. becomes

$$\therefore \frac{dz}{dx} + Pz = Q$$

which is linear

~~Eg~~ $\cancel{x\frac{dy}{dx} + 3y = x^4 e^{1/x} y^3}$

eq 4

$\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

$\div$ by $\cos y$ $\left(\cancel{\text{Sep} \int \tan y \, dy = \sec y} \right) \therefore Q = f(x)$ only

$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x$

Let $\sec y = z$

$\therefore \frac{dz}{dx} = \sec y \tan y \frac{dy}{dx}$

$\therefore \frac{dz}{dx} + z \tan x = \cos^2 x$

$\because$ The eq is linear $\therefore$ ~~its soln is~~

$IF = e^{\int P dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

$\therefore$ Soln is

$z(IF) = c + \int Q(IF) dx$

$z \sec x = c + \int \cos^2 x \sec x \, dx$

$z \sec x = c + \int \cos x \, dx$

$z \sec x = c + \sin x$

$\sec y \sec x = c + \sin x$

Ex. II-f

1 $x \frac{dy}{dx} + 3y = x^4 e^{1/x} y^3$

$\div x y^3$

$\therefore y^{-3} \frac{dy}{dx} + 3y^{-2} . x^{-1} = x^3 e^{1/x}$

Let $y^{-2} = z \quad \therefore -2y^{-3} \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore y^{-3} \frac{dy}{dx} = -\frac{1}{2} \frac{dz}{dx}$

$-\frac{1}{2} \frac{dz}{dx} + 3z . x^{-1} = x^3 e^{1/x}$

$\therefore \frac{dz}{dz} - 6z . x^{-1} = -2x^3 e^{1/x}$

Is is linear eq.

$P = -6x^{-1}, \quad Q = 2x^3 e^{1/x}$

$IF = e^{\int P\,dx} = e^{\int -6x^{-1}dx} = e^{-6\int \frac{dx}{x}}$

$= e^{-6\log x} = x^{-6}$

Soln. is

~~$z(IF) = c + \int Q(IF)\,dx$~~

~~$ze^{-3x^2} = c + \int x^3 e^{1/x} \cdot e^{-3x^2} dx$~~

~~$ze^{-3x^2} = c + \int x^3 e^{\frac{1}{x}-3x^2} dx$~~

$z(IF) = c + \int Q(IF)\,dx$

$zx^{-6} = c + \int 2x^3 e^{1/x} x^{-6} dx$

$zx^{-6} = c + 2\int e^{1/x} x^{-3} dx$

$= c + 2\int e^{1/x} \left(\frac{1}{x^2}\right) dx \cdot \frac{1}{x}$

Let $\frac{1}{x} = t \quad \therefore -\frac{1}{x^2}dx = dt$

$\int Q \cdot x^{-6} dx = e^{1/x^2} \quad \therefore Qx^{-6} = e^{1/x^2} \cdot -2/x^3 = 2e^{1/x^2} x^{-3}$

$zx^{-6} = c + 2\int e^t \cdot t (-dt)$

$zx^{-6} = c + 2\int t e^t dt$

$zx^{-6} = c + 2[te^t - e^t]$

$zx^{-6} = c + \frac{2}{x} e^{1/x} - 2e^{1/x}$

$\frac{1}{y^2 x^6} = c + \frac{2}{x} e^{1/x} - 2e^{1/x}$

Tel: 9820246760 / 9769479368 / 9820009639

2. $\frac{dy}{dx} + x^2 y = x^5$

$\because$ Eq is linear

$\therefore P = x^2$; $Q = x^5$

$IF = e^{\int P dx} = e^{\int x^2 dx} = e^{\frac{x^3}{3}}$

Soln is

$y(IF) = c + \int Q(IF)dx$

$ye^{x^3/3} = c + \int x^5 . e^{x^3/3} dx$

Let $\frac{x^3}{3} = t \quad \therefore \frac{3x^2}{3} dx = dt \; ; \; x^3 = 3t$

$ye^{x^3/3} = c + \int 3te^t dt$

$ye^{x^3/3} = c + 3te^t - 3e^t$

$ye^{x^3/3} = c + x^3 e^{x^3/3} - 3e^{x^3/3}$

$ye^{x^3/3} - x^3 e^{x^3/3} + 3e^{x^3/3} = C$

$e^{x^3/3}(y - x^3 + 3) = C$

3. $3y^2 \frac{dy}{dx} + 2xy^3 = 4xe^{-x^2}$

Let $y^3 = t \quad \therefore 3y^2 \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore \frac{dt}{dx} + 2xt = 4xe^{-x^2}$ is a linear eq.

$P = 2x$, $Q = 4xe^{-x^2}$, $IF = e^{\int 2x dx} = e^{x^2}$

$\therefore$ Soln is

$t(IF) = c + \int Q(IF)dx$

$y^3 e^{x^2} = c + \int 4x\cancel{e^{-x^2}} . \cancel{e^{x^2}} dx$

$y^3 e^{x^2} = c + 2x^2$

4. $(1+x^2)dy = (\tan^{-1}x - y)dx$

$(1+x^2)\frac{dy}{dx} + y = \tan^{-1}x \quad (\div \text{ by } 1+x^2)$

$\therefore \frac{dy}{dx} + \frac{1}{1+x^2} y = \frac{\tan^{-1}x}{1+x^2}$

Tel: 9820246760 / 9769479368 / 9820009639

navlakhi.com navlakhi.mobi navlakhi.tv

Is a linear eq

$\therefore P = \frac{1}{1+x^2}$; $Q = \frac{\tan^{-1}x}{1+x^2}$

$IF = e^{\int P dx} = e^{\int \frac{dx}{1+x^2}} = e^{\tan^{-1}x}$

$y\,IF = c + \int Q(IF)\,dx$

$y e^{\tan^{-1}x} = c + \int \frac{\tan^{-1}x}{1+x^2} e^{\tan^{-1}x} dx$

Let $\tan^{-1}x = t$ $\therefore \frac{1}{1+x^2} dx = dt$

$y e^{\tan^{-1}x} = c + \int t e^t dt$

$= c + te^t - e^t$

$y e^{\tan^{-1}x} = c + \tan^{-1}x\, e^{\tan^{-1}x} - e^{\tan^{-1}x}$

$(\div$ by $e^{\tan^{-1}x})$

$y = ce^{-\tan^{-1}x} + \tan^{-1}x - 1$

5. $3x(1-x^2)y^2\frac{dy}{dx} + (2x^2-1)y^3 = ax^3$

$y^2\frac{dy}{dx} + \frac{2x^2-1}{3x(1-x^2)}y^3 = \frac{ax^3}{3x(1-x^2)}$

Let $y^3 = z$

$\therefore 3y^2\frac{dy}{dx} = \frac{dz}{dx}$

$y^2\frac{dy}{dx} = \frac{1}{3}\frac{dz}{dx}$

$\frac{1}{3}\frac{dz}{dx} + \frac{2x^2-1}{3x(1-x^2)} z = \frac{ax^3}{3x(1-x^2)}$

$\therefore \frac{dz}{dx} + \frac{2x^2-1}{x(1-x^2)} = \frac{ax^3}{x(1-x^2)}$.

Is a linear eq.

$If = e^{\int P dx} = e^{\int \frac{2x^2-1}{x(1-x^2)}}$

$I = \int \frac{2x^2 dx}{x(1-x^2)} - \frac{1}{2}\int \frac{2x}{x(1-x^2)} dx$

Let $1+x^2=t$

$\therefore -2x\,dx = dt$

$I = -\int \frac{dt}{t} + \frac{1}{2}\int \frac{dt}{t(1-t)}$

$= -\log t - \frac{1}{2}\int \frac{dt}{t^2-t+\frac{1}{4}-\frac{1}{4}}$

$= -\log t - \frac{1}{2}\int \frac{dt}{(t-\frac{1}{2})^2-(\frac{1}{2})^2}$

$= -\log t - \frac{1}{2}\times\frac{1}{2\times\frac{1}{2}}\log\left(\frac{t-\frac{1}{2}-\frac{1}{2}}{t-\frac{1}{2}+\frac{1}{2}}\right)$

$= -\log t - \frac{1}{2}\log\left(\frac{t-1}{t}\right)$

$= -\log t \cdot \sqrt{\frac{t-1}{t}} = -\log\sqrt{t(t-1)}$

$= -\log\sqrt{(1-x^2)(-x^2)} = -\log x\sqrt{(x^2-1)}$

$IF = e^{\int P\,dx} = e^{\log\frac{1}{x\sqrt{x^2-1}}} = \frac{1}{x\sqrt{x^2-1}}$

$1-x^2$

Soln. (i)

$z(IF) = c + \int Q(IF)\,dx$

$\frac{z}{x\sqrt{x^2-1}} = c + \int \frac{ax^2}{(1-x^2)}\cdot\frac{1}{x\sqrt{x^2-1}}\,dx$

$\frac{z}{x\sqrt{x^2-1}} = c - \int \frac{ax}{(x^2-1)^{3/2}}\,dx$

Let $x^2-1=t$

$\therefore 2x\,dx = dt$

$\therefore x\,dx = dt/2$

$\frac{z}{x\sqrt{x^2-1}} = c - \frac{a}{2}\int \frac{dt}{t^{3/2}}$

$\frac{z}{x\sqrt{x^2-1}} = c + \frac{a}{2}\frac{t^{-1/2}}{\frac{1}{2}}$

$\frac{z}{x\sqrt{x^2-1}} = c + \frac{a}{\sqrt{x^2-1}}$

Tel: 9820246760 / 9769479368 / 9820009639

navlakhi.com navlakhi.mobi navlakhi.tv

$z = cx\sqrt{x^2-1} + ax$

$y^3 = cx\sqrt{x^2-1} + ax$

6 $x\,dy - \{y + xy^3(1+\log x)\}dx = 0$

$x\frac{dy}{dx} - y - xy^3(1+\log x) = 0$

$x\frac{dy}{dx} - y = y^3 x(1+\log x)$

$\div\ xy^3$

$y^{-3}\frac{dy}{dx} - \frac{y^{-2}}{x} = (1+\log x)$

Let $y^{-2} = z$

$\therefore -2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$

$y^{-3}\frac{dy}{dx} = \frac{-1}{2}\frac{dz}{dx}$

$-\frac{1}{2}\frac{dz}{dx} - \frac{z}{x} = (1+\log x)$

$\frac{dz}{dx} + \frac{2}{x}z = -2(1+\log x)$

is linear eq

$IF = e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{\log x^2} = x^2$

Soln. is

$z(IF) = c + \int Q(IF)\,dx$

$zx^2 = c + \int -2(1+\log x)\,x^2\,dx$

$zx^2 = c - 2\left[\int x^2 dx + \int x^2 \log x\,dx\right]$

$zx^2 = c - 2\left[\frac{x^3}{3} + \log x\cdot\frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}dx\right]$

$zx^2 = c - 2\left[\frac{x^3}{3} + \log x\cdot\frac{x^3}{3} - \frac{x^3}{9}\right]$

$\frac{x^2}{y^2} = c - 2\left[\frac{2x^3}{9} + \frac{x^3}{3}\log x\right]$

$$\frac{x^2}{y^2} = -\frac{2}{3}x^3\left(\frac{2}{3} + \log x\right) + C$$

7 $\frac{dy}{dx}\cos x + y\sin x = (y\sec x)^{1/2}$

$\cos x \frac{dy}{dx} + y\sin x = y^{1/2}\sec^{1/2} x$

$\div\ y^{1/2}\cos x$

$y^{-1/2}\frac{dy}{dx} + y^{1/2}\tan x = \frac{\sec^{1/2} x}{\cos x}$

Let $y^{1/2} = z$

$\frac{1}{2}y^{-1/2}\frac{dy}{dx} = \frac{dz}{dx}$

$y^{-1/2}\frac{dy}{dx} = 2\frac{dz}{dx}$

$2\frac{dz}{dx} + z\tan x = \sec^{3/2} x$

$\frac{dz}{dx} + \frac{\tan x}{2}.z = \frac{\sec^{3/2} x}{2}$

Is a linear eq ∴

$\therefore IF = e^{\int P dx} = e^{\int \frac{\tan x}{2}} = e^{\log \sqrt{\sec x}} = \sqrt{\sec x}$

$z(IF) = c + \int Q(IF)\,dx$

$z\sqrt{\sec x} = c + \int \frac{\sec^{3/2} x . \sec^{1/2} x}{2}\,dx$

$z\sqrt{\sec x} = c + \frac{1}{2}\int \sec^2 x\,dx$

$z\sqrt{\sec x} = c + \frac{1}{2}\tan x$

$2z\sqrt{\sec x} = c' + \tan x \quad (c' = 2c)$

$2\sqrt{y\sec x} = c' + \tan x$

Sq

$4y\sec x = (c' + \tan x)^2$

$4y = (c' + \tan x)^2 \cos x$

8. $e^{-y}\sec^2 y\,dy = dx + x\,dy$

$(e^{-y}\sec^2 y - x)\,dy = dx$

$\frac{dx}{dy} = -x + e^{-y}\sec^2 y$

$\frac{dx}{dy} + x = e^{-y}\sec^2 y$

is a linear eq

$\therefore$ IF $= e^{\int P\,dy} = e^{\int 1\,dy} = e^{y}$

$\therefore$ Soln is

$x(IF) = c + \int Q(IF)\,dy$

$xe^{y} = c + \int e^{-y}\sec^2 y\, e^{y}\,dy$

$xe^{y} = c + \tan y$

9. $(1-x^2)\frac{dy}{dx} = xy(1+x^2y^2)$

$(1-x^2)\frac{dy}{dx} - xy = x^3y^3$

$\div\ y^3(1-x^2)$

$y^{-3}\frac{dy}{dx} - y^{-2}\left(\frac{x}{1-x^2}\right) = \frac{x^3}{1-x^2}$

Let $y^{-2} = z$

$\therefore -2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}$

$-\frac{1}{2}\frac{dz}{dx} - \frac{x}{1-x^2}\cdot z = \frac{x^3}{1-x^2}$

$\frac{dz}{dx} + \frac{2x}{1+x^2}\cdot z = -\frac{2x^3}{1+x^2}$

navlakhi.com navlakhi.mobi navlakhi.tv

Tel: 9820246760 / 9769479368 / 9820009639

Linear eq. in z

$$IF = e^{\int P dx} = e^{-\int \frac{-2x}{1-x^2} dx} = e^{\log \frac{1}{1-x^2}}$$

$$= \frac{1}{1-x^2}$$

Soln. is

$$z(IF) = c + \int Q(IF)\,dx$$

$$z\left(\frac{1}{1-x^2}\right) = c + \int -\frac{2x^3}{1-x^2} \cdot \frac{1}{1-x^2}\,dx$$

Let $1-x^2 = t \quad \therefore -2x\,dx = dt$

$x^2 = 1-t$

$$\frac{z}{1-x^2} = c + \int \frac{(1-t)\,dt}{t^2}$$

$$\frac{y^{-2}}{1-x^2} = c + \int \frac{1}{t^2}\,dt - \int \frac{dt}{t}$$

$$\frac{y^{-2}}{1-x^2} = c - \frac{1}{t} - \log t$$

$$\frac{y^{-2}}{1-x^2} = c - \frac{1}{1-x^2} - \log(1-x^2)$$

$$\frac{1}{y^2(1-x^2)} = c - \frac{1}{1-x^2} - \log(1-x^2)$$

$$1 = cy^2(1-x^2) - y^2 - y^2(1-x^2)\log(1-x^2)$$

$$y^2 + 1 = (1-x^2)y^2\left[c - \log(1-x^2)\right]$$

10 $\quad x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$

$$\frac{dy}{dx} - \frac{x-2}{x(x-1)}\,y = \frac{x^2(2x-1)}{x-1}$$

Is a linear eq.

$$\therefore IF = e^{\int P dx} = e^{-\int \frac{x-1-1}{x(x-1)}}$$

$$I = -\int \frac{x-1}{x(x-1)}dx + \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}}\,dx$$

$$= -\log x + \int \frac{dx}{(x-\frac{1}{2})^2 - (\frac{1}{2})^2}$$

$= -\log x + \frac{1}{2\times\frac{1}{2}}\log\left(\frac{x-\frac{1}{2}-\frac{1}{2}}{x-\frac{1}{2}+\frac{1}{2}}\right)$

$= -\log x + \log\left(\frac{x-1}{x}\right) = \log\left(\frac{x-1}{x^2}\right)$

$IF = e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$

Soln. is

$y(IF) = c + \int Q(IF)\,dx$

$\frac{y}{x^2}(x-1) = c + \int \frac{x^2(2x-1)}{x-1} \times \frac{x-1}{x^2}\,dx$

$\frac{y}{x^2}(x-1) = c + \frac{2x^2}{2} - x$

$y = \frac{(c+x^2-x)x^2}{x-1}$

4 $x^2\frac{dy}{dx} = 3x^2 - 2xy + 1$

$x^2\frac{dy}{dx} + 2xy = 3x^2 + 1$

$\div$ by x^2

$\frac{dy}{dx} + \frac{2y}{x} = \frac{3x^2+1}{x^2}$

is linear eq

$IF = e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{\log x^2} = x^2$

Soln. is

$y(IF) = c + \int Q(IF)\,dx$

$yx^2 = c + \int \frac{3x^2+1}{x^2} \cdot x^2\,dx$

$yx^2 = c + \frac{3x^3}{3} + x$

$\therefore y = \frac{c}{x^2} + x + \frac{1}{x}$

METHOD VII: SUBSTITUTION:

Type 1: Eq. of the type

$$\frac{dy}{dx} = f(ax+by+c)$$

we substitute $u = ax+by+c$

Eg. $(x-y)^2 \frac{dy}{dx} = a^2$

Let $u = x-y$

$\therefore \frac{du}{dx} = 1 - \frac{dy}{dx}$

$\therefore \frac{dy}{dx} = 1 - \frac{du}{dx}$

$\therefore u^2\left(1 - \frac{du}{dx}\right) = a^2$

$u^2 - u^2\frac{du}{dx} = a^2$

$\therefore u^2 \frac{du}{dx} = u^2 - a^2$

$\therefore \int \frac{u^2 - a^2 + a^2}{u^2-a^2} du = \int dx$

$\int du + a^2 \int \frac{du}{u^2-a^2} = x + c$

$u + a^2 \cdot \frac{1}{2a} \log\left(\frac{u-a}{u+a}\right) = x + c$

$\cancel{x} - y + \frac{a}{2}\log\left(\frac{x-y-a}{x-y+a}\right) = \cancel{x} + c$

$\therefore \frac{a}{2}\log\left(\frac{x-y-a}{x-y+a}\right) = y + c.$

Type 2: If the D.E involves as a factor $(x\,dy - y\,dx)$, substitute $v = y/x$

$\therefore x^2 dv = x\,dy - y\,dx$

If the DE involves as a factor $(x\,dx + y\,dy)$, substitute $u = x^2+y^2$

$\therefore du/2 = xdx + ydy$

If the DE involves both the factors $(xdx + ydy)$ & $(xdy - ydx)$, then substitute $x = r\cos\theta$, $y = r\sin\theta$

$\therefore xdx + ydy = rdr$

$xdy - ydx = r^2 d\theta$

$r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}(y/x)$

Eg

$xdx + ydy = m(xdy - ydx)$

Subs. $x = r\cos\theta$, $y = r\sin\theta$

$\therefore xdx + ydy = rdr$,

$xdy - ydx = r^2 d\theta$

$\therefore rdr = mr^2 d\theta$

$\frac{dr}{r} = m\,d\theta$

$\therefore \log r = m\theta + c$

$\therefore \log\sqrt{x^2+y^2} = m\tan^{-1}\frac{y}{x} + c$

$\log(x^2+y^2) = 2m\tan^{-1}\frac{y}{x} + c'$

Type 3: Changing dependent & indepent var. to get the eq. in standard form eg. Bernoulli eq.

Eg

$(x^2y^3 + xy)dy = dx$

$\frac{dx}{dy} = yx + y^3x^2$

$\therefore \frac{dx}{dy} - yx = y^3x^2$

$\div$ by x^2

$-x^{-2}\frac{dx}{dy} + yx^{-1} = -y^3$

Let $x^{-1} = z$ $\therefore -x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$

$\frac{dz}{dy} + yz = -y^3$

$IF = e^{\int P dy} = e^{\int y dy} = e^{\frac{y^2}{2}}$

Soln. is

$z(\text{IF}) = c + \int Q(\text{IF})\,dy$

$z(e^{y^2/2}) = c + \int -y^3 e^{y^2/2}\,dy$

Let $\frac{y^2}{2} = t \quad \therefore y\,dy = dt \; ; \; y^2 = 2t$

$\frac{e^{y^2/2}}{x} = c - 2\int t e^t\,dt$

$\frac{e^{y^2/2}}{x} = c - 2\left[te^t - e^t\right]$

$\frac{e^{y^2/2}}{x} = c - 2\frac{y^2}{2} e^{y^2/2} + 2e^{y^2/2}$

$e^{y^2/2} = cx - xy^2 e^{y^2/2} + 2x e^{y^2/2}$

Type 4: Subs. of terms

eg) $(2 + 2x^2 y^{1/2})y\,dx + (x^2 y^{1/2} + 2)x\,dy = 0$

Let $x^2 y^{1/2} = v \quad \therefore y = \frac{v^2}{x^4}$

$\cancel{2xy^{1/2}dx + \frac{1}{2}x^2 y^{-1/2}dy = dv}$

$dy = \frac{2v}{x^4}dv - \frac{4v^2}{x^5}dx$

$\therefore \left(2 + 2x^2\frac{v}{x^2}\right)\frac{v^2}{x^4}dx + \left(x^2\frac{v}{x^2} + 2\right)x\left(\frac{2v\,dv}{x^4} - \frac{4v^2}{x^5}dx\right) = 0$

$(2 + 2v)\frac{v^2}{x^4}dx + (v+2)x\left(\frac{2vx\,dv - 4v^2 dx}{x^5}\right) = 0$

$2(1+v)v\,dx + 2(v+2)(x\,dv - 2v\,dx) = 0$

$\cancel{v^2 + v + 2v}$

$\{v\,dx + v^2 dx + xv\,dv - 2v^2 dx + 2x\,dv - 4v\,dx = 0$

$(-v^2 - 3v)dx + (xv + 2x)dv = 0$

$-v(v+3)dx + x(v+2)dv = 0$

$\div\; xv(v+3)$

$-\int \frac{dx}{x} + \int \frac{v+2}{v(v+3)}dv = 0$

$\frac{v+2}{v(v+3)} = \frac{A}{v} + \frac{B}{v+3} = \frac{A(v+3) + Bv}{v(v+3)}$

$$-\int \frac{dx}{x} + \int \frac{2\,dv}{3v} + \frac{1}{3}\int \frac{dv}{v+3} = C$$

$$-\log x + \frac{2}{3}\log v + \frac{1}{3}\log(v+3) = \log c$$

$$-3\log x + 2\log v + \log(v+3) = \log c$$

$$2\log v + \log(v+3) = 3\log x + \log c$$

$$\log v^2(v+3) = \log x^3 c$$

$$x^4 y(x^2\sqrt{y}+3) = x^3 C$$

$$xy(x^2\sqrt{y}+3) = C$$

Eg 2 $(2x^2+3y^2-7)x\,dx - (3x^2+2y^2-8)y\,dy$

Let $x^2 = u \therefore x\,dx = du/2$

$y^2 = v \therefore y\,dy = dv/2$

$$(2u+3v-7)du - (3u+2v-8)dv = 0$$

$$\frac{du}{dv} = \frac{3u+2v-8}{2u+3v-7}$$ Is a non-homog. linear eq.

$u = U+h;\ v = V+k$

$$\frac{dU}{dV} = \frac{3U+2V+(3h+2k-8)}{2U+3V+(2h+3k-7)}$$

$3h+2k=8$ —(I) $h+k=3$

$2h+3k=7$ —(II) $h-k=1$

$\cancel{h=1;\ k=5/2}$ $h=2;\ k=1$

$$\therefore \frac{dU}{dV} = \frac{3U+2V}{2U+3V}$$ Is a homog. eq.

Let $U = AV$

$$\frac{dU}{dV} = A + V\frac{dA}{dV}$$

$$A + V\frac{dA}{dV} = \frac{3AV+2V}{2AV+3V} = \frac{3A+2}{2A+3}$$

$$V\frac{dA}{dV} = \frac{3A+2}{2A+3} - A = \frac{3A+2-2A^2-3A}{2A+3}$$